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Classifying rendezvous tasks of arbitrary dimension

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ABSTRACT

The rendezvous is a type of distributed decision tasks including many well-known tasks such as set agreement, simplex agreement, and approximation agreement. An n-dimensional rendezvous task, $n \geq 1$, allows $n + 2$ distinct input values, and each execution produces at most $n + 2$ distinct output values. A rendezvous task is said to implement another if an instance of its solution, followed by a protocol based on shared read/write registers, solves the other. The notion of implementation induces a classification of rendezvous tasks of every dimension: two tasks belong to the same class if they implement each other. Previous work on classifying rendezvous tasks only focused on 1-dimensional ones.

This paper solves an open problem by presenting the classification of nice rendezvous of arbitrary dimension. An n-dimensional rendezvous task is said to be *nice* if the qth reduced homology group of its decision space is trivial for $q \neq n$, and free for $q = n$. Well-known examples are set agreement, simplex agreement, and approximation agreement. Each n-dimensional rendezvous task is assigned an algebraic signature, which consists of the nth homology group of the decision space, as well as a distinguished element in the group. It is shown that an n-dimensional nice rendezvous task implements another if and only if there is a homomorphism from its signature to that of the other. Hence the computational power of a nice rendezvous task is completely characterized by its signature.

In each dimension, there are infinitely many classes of rendezvous tasks, and exactly countable classes of nice ones. A representative is explicitly constructed for each class of nice rendezvous tasks.

1. Introduction

A distributed computing system consists of finitely many sequential processes communicating via shared read/write registers and other mechanisms [16]. The mechanisms include communication channels, synchronizing primitives, and general services [1,9]. The processes are asynchronous and may fail by stopping, so it is indistinguishable whether an irresponsive process has failed or is only running slowly. A protocol is a distributed program in such a system. A task is a distributed coordination problem where each process starts with a private input value and decides an output value such that the decisions of all processes meet some specification [10]. Well-known examples of tasks include consensus [8], set consensus[5], and renaming [2]. A protocol is said to solve a task if, starting with any legal input assignment, the outputs produced in any execution of the protocol meet the task specification.

This paper focuses on rendezvous tasks [11,15], which intuitively model the scenarios where autonomous agents move around in a specific space to meet one another. The significance of rendezvous lies in three aspects. First, it can be used in many applications, for example web-crawling, peer-to-peer lookup, and meeting scheduling. Second, it includes many

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well-known tasks such as set agreement and approximation agreement, so the research will provide an underlying theory and systematic methods for these tasks. Third, it plays a critical role in proving the undecidability of a variety of distributed tasks [10].

Computability and efficiency [3,7,12] are important topics on rendezvous tasks. Work on these topics (for example [14, 17]), except [10,11], mainly considered specific rendezvous tasks, and not general theory or systematic methods.

In [10], Herlihy and Rajsbaum investigated the computability of loop agreement — a type of rendezvous tasks. A loop agreement task is defined in terms of an edge loop in a 2-complex, with three distinguished points on the loop. It stands for a task with the distinguished points as input values and the vertices of the 2-complex as output values. In an execution, if the inputs are the same, the outputs all coincide with the input; if the inputs have two distinct values, the outputs span a simplex along the segment of the loop connecting the two points; otherwise, the outputs span an arbitrary simplex in the complex. [10] showed that a loop agreement task is solvable in certain models if and only if the loop is contractible in the 2-complex, so the solvability of loop agreement tasks in these models is undecidable.

In [11], a classification of loop agreement tasks was presented based on their relative computability. It considered whether a task T_1 can implement T_2, i.e. T_2 can be solved by calling an instance of a solution to T_1, followed by a protocol based on shared read/write registers. Loop agreement tasks can be classified according to the equivalence relation induced by implementation. [11] assigned an algebraic signature to each loop agreement task, which is a pair consisting of the fundamental group of the 2-complex and the path class represented by the loop. It was shown that T_1 can implement T_2 if and only if there is a homomorphism from the signature of T_1 to that of T_2. As a result, the signature completely characterizes the computability of a loop agreement task.

[10,11] only considered loop agreement. We call loop agreement the 1-rendezvous (rendezvous of dimension 1), since a loop is a 1-dimensional topological space. Any task with more than three input values cannot be cast as a 1-rendezvous task, including the well-known $(n+1, n)$-agreement[6] for $n \geq 3$. In addition, the classification in [11] was not constructive, in the sense that a representative was not constructed for each class. Hence, this paper explores an open problem proposed by [11], trying to extend the results in [11] from dimension 1 to arbitrary dimension.

Surprisingly, this seemingly natural extension is still pending, possibly due to the following obstacle. The power of the signatures of 1-rendezvous tasks comes from the fact that any homomorphism between signatures is induced by a continuous map between the 2-complexes. But this fails in higher dimensions, no matter whether signatures are defined in terms of homotopy groups or homology groups. Hence, the "if" part of the main result in [11] does not hold generally in higher-dimensional cases. Proper constraints must be imposed on the complexes in order to guarantee the power of signatures.

This paper defines an n-dimensional rendezvous task, or n-rendezvous task, in terms of an $(n+1)$-complex and a simplicial embedding of a subdivided n-sphere to the complex. The $(n+1)$-complex is called its decision space. Given a generator of the nth homology group of the n-sphere, the embedding uniquely determines an element in the nth homology group of the decision space, so the definition of signatures of 1-rendezvous can be adapted to the n-dimensional case. To retain the power of the signatures, we require the decision space to be simply connected, with mth reduced homology group trivial for $m \neq n$, and free Abelian otherwise. A rendezvous task satisfying this constraint is said to be *nice*. The niceness property enables any signature homomorphism to be induced by a continuous map.

The main contribution lies in the following aspects.

- For rendezvous tasks of arbitrary dimension, their algebraic signatures are defined. We show that, generally, one rendezvous task implements another only if there is a homomorphism from its signature to that of the other, and vice versa if the tasks are nice. So, the signatures of nice rendezvous tasks completely characterize their computability.
- The nice rendezvous tasks are divided into infinitely many, countable classes, according to the equivalence relation determined by mutual implementation.
- A representative is constructed for each class of nice rendezvous tasks.

The rest of this paper is organized as follows. In Section 2, preliminaries on topology and distributed tasks are presented. In Section 3, rendezvous tasks and their algebraic signatures are defined. Sections 4 and 5 respectively deal with the necessary and sufficient conditions of implementing one task from another. Section 6 provides further observations based on our main results in Section 5. Section 7 concludes this paper.

2. Preliminaries

This section will introduce our distributed computing model and formalize the notion of a task. Necessary material from algebraic topology is also presented, because our main techniques come from homology theory and some from homotopy theory. There is a long line of work in distributed computability (for example, [4,13,18]) also borrowing tools from algebraic topology.

2.1. System model and task formalization

The computing model and task formalization coincide with those in [11], so we will present them very briefly. Interested readers should please refer to Subsection 3.1 of [11].

We adopt the shared-memory model [16] for distributed computing, where a system consists of a finite set of asynchronous sequential processes, which communicate through shared memory. The shared memory includes read/write registers and possibly more powerful objects and services. A process may delay indefinitely, or fail by stopping.

A task is a distributed coordination problem in which each process starts with a private input value, communicates with others via shared memory, produces an output value, and halts.

Formally, an m-process task T is specified by a triple $(\mathcal{I}, \mathcal{O}, \Delta)$, where $\mathcal{I} \subseteq (D_{\mathbb{I}} \bigcup \{\perp\})^m \setminus \{(\perp, \ldots, \perp)\}$ is the set of input vectors, $\mathcal{O} \subseteq (D_{\mathbb{O}} \bigcup \{\perp\})^m \setminus \{(\perp, \ldots \perp)\}$ is the set of output vectors, and $\Delta \subseteq \mathcal{I} \times \mathcal{O}$ is the task specification. $D_{\mathbb{I}}$ and $D_{\mathbb{O}}$ are the input and output data types, respectively. $\mathcal{I}$ and $\mathcal{O}$ are both prefix-closed [11]. An element $I \in \mathcal{I}$ represents an assignment of input values in an execution: if $I_i \neq \perp$, the ith process starts with input I_i, otherwise it does not participate in that execution. The meaning of output vectors can be likewise understood. Δ carries an input vector to a set of matching output vectors, specifying the *legal* outputs for that input assignment. Here, vectors $I \in \mathcal{I}$ and $O \in \mathcal{O}$ are said to match when, for any i, $I_i = \perp$ if and only if $O_i = \perp$.

An m-process protocol is said to r-resiliently solve a task $(\mathcal{I}, \mathcal{O}, \Delta)$ if, for every execution where the input vector is I and at least $n - r$ processes decide, the decision vector is a prefix of some output vector in $\Delta(I)$. When $r = n - 1$, the protocol is said to be wait-free.

We also borrow the notion of implementation from [11]. A task T is said to be implementable from task T' if T can be solved by calling an instance of a protocol that solves T', possibly followed by accessing shared read/write registers. Implementation naturally induces an equivalence relation where two tasks are equivalent if and only if they are mutually implementable.

2.2. Some concepts and facts in algebraic topology

We recall necessary preliminaries in algebraic topology. For further information, please refer to [19,20].

2.2.1. Simplicial complexes and quotient spaces

For $n \geq 0$, the standard n-sphere is the subspace $\mathcal{S}^n = \left\{(t_0, t_1, \ldots, t_n) \in R^{n+1} | \sum_{i=0}^n t_i^2 = 1\right\}$ of $(n+1)$-dimensional Euclidean space R^{n+1}.

Arbitrarily choose a finite set of points $\{v_0, v_1, \ldots, v_d\} \subset R^n$. If they are affinely independent, the convex closure $s = \left\{\sum_{i=0}^d \lambda_i v_i | \sum_{i=0}^d \lambda_i = 1 \text{ and each } \lambda_i \geq 0\right\}$ is called the simplex spanned by $\{v_0, v_1, \ldots, v_d\}$, and is denoted by $\overline{\{v_0, v_1, \ldots, v_d\}}$. The simplex spanned by any subset of $\{v_0, v_1, \ldots, v_d\}$ is called a face of s. Each v_i is called a vertex of s. The dimension of s is defined to be d.

A finite set of well-positioned simplexes in a Euclidean space, together with all their faces, is called a (simplicial) complex. A complex is said to be an n-complex if each simplex in it is of dimension no more than n. A complex K' is said to be a subcomplex of K if $K' \subseteq K$.

A map f from complex K to K' is simplicial if, for each vertex v of K, $f(v)$ is also a vertex of K', and, for each simplex $s = \overline{\{v_0, v_1, \ldots, v_d\}} \in K$, $f(s)$ is spanned by the set of $f(v_0), f(v_1), \ldots, f(v_d)$. An injective simplicial map is called a simplicial embedding.

For any complex K in R^n, the subspace $|K| = \bigcup_{s \in K} s$ of R^n is called the polyhedron of K. A simplicial map $f : K \to K'$ induces a continuously map $|f| : |K| \to |K'|$, $|f|(v) = \sum_{i=0}^d \lambda_i f(v_i)$ for each $v = \sum_{i=0}^d \lambda_i v_i \in \overline{\{v_0, v_1, \ldots, v_d\}} \in K$.

A complex $\sigma(K)$ is called a subdivision of complex K if any simplex in $\sigma(K)$ lies in a simplex in K, and any simplex in K is the union of some simplexes in $\sigma(K)$.

Given topological spaces X, Y and a continuous map $f : A \subseteq X \to Y$, let $\mathcal{R}_f$ be the smallest equivalence relation over $X \bigsqcup Y$ such that x and $f(x)$ are equivalent for $x \in A$, where $X \bigsqcup Y$ is the disjoint union of X and Y. Endow $(X \bigsqcup Y)/\mathcal{R}_f$ with the most refined topology such that the quotient map $\rho : X \bigsqcup Y \to (X \bigsqcup Y)/\mathcal{R}_f$ is continuous. Then the topological space $(X \bigsqcup Y)/\mathcal{R}_f$ is called the quotient space of X and Y under f, and simply denoted $(X \bigsqcup Y)/f$, or X/f when f is surjective.

Example 1. Let $X = \mathcal{S}^n \times \{1, 2, \ldots, r\}$, $*$ be the singleton space, $A = \{v_0\} \times \{1, 2, \ldots, r\}$, where $v_0 = (1, 0, \ldots, 0) \in \mathcal{S}^n$, and $f : A \to *$. The quotient space $W_r^n = X/f$ is called the wedge sum of r n-spheres. Generally, given topological spaces X_i, $1 \leq i \leq m$, and $A = \{v_0, \ldots, v_r\}$ with $v_i \in X_i$ for each i, then the quotient space $(\bigsqcup_{i=1}^m X_i)/f$, where $f : A \to *$, is called the wedge sum of X_i's.

Fig. 1 is an illustration of the wedge sum of r 2-spheres.

2.2.2. Homotopy

Consider topological spaces X, Y with subsets $A \subseteq X$ and $B \subseteq Y$, and two continuous maps $f, g : X \to Y$. Let $[0, 1]$ stand for the segment from 0 to 1 on the real line. If there is a continuous map $H : [0, 1] \times X \to Y$ such that $H(0, \cdot) = f$ and $H(1, \cdot) = g$, then f and g are homotopic, denoted $f \sim_H g$ or simply $f \sim g$. When $H([0, 1] \times A) \subseteq B$, f and g are homotopic relative to A and B, denoted $f \sim_H g$ *rel* A, B or $f \sim_{A,B} g$. If there is a map $h : Y \to X$ such that $hf \sim id_X$ and $fh \sim id_Y$, then X and Y are said to be homotopy equivalent, denoted $X \simeq Y$, and f is called a homotopy equivalence. Hereunder, the symbol id_X denotes the identity map on space X.

Both homotopy and relative homotopy are equivalence relations.

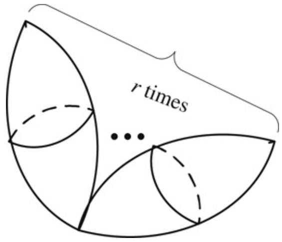

Fig. 1. The wedge sum of r 2-spheres.

(Homotopic Extension Theorem). *Given complexes X, Y, and a subcomplex $A \subseteq X$, if maps $f : X \to Y$ and $g : A \to Y$ satisfy that $f|_A \sim_H g$, then there is $H' : [0, 1] \times X \to Y$ such that $H'(0, \cdot) = f$, $H'(1, \cdot)|_A = g$, and $H'|_{[0,1]\times A} = H$.*

Let $\pi_n(X, x_0) = \{f | f : (\mathcal{S}^n, v_0) \to (X, x_0)\} / \sim_{v_0,x_0}$, where $v_0 = (1, 0, \dots, 0) \in \mathcal{S}^n$, $x_0 \in X$. There is a canonical binary operation "+" such that $(\pi_n(X, x_0), +)$ is a group, which is still denoted by $\pi_n(X, x_0)$ and is Abelian when $n > 1$. For a path-connected space X, different choices of x_0 always produce isomorphic $\pi_n(X, x_0)$, so it is justified to write $\pi_n(X)$ for $\pi_n(X, x_0)$ for such X. Hereunder, given $f : (\mathcal{S}^n, v_0) \to (X, x_0)$, the symbol $[f]$ denotes the element of $\pi_n(X, x_0)$ that contains f.

Example 2. For $n > 0$, $\pi_n(\mathcal{S}^n)$ is isomorphic to the integer group Z, generated by $[id_{\mathcal{S}^n}]$.

A map $f : (X, x_0) \to (Y, y_0)$ induces a homomorphism $f_n^\sharp : \pi_n(X, x_0) \to \pi_n(Y, y_0)$, $[g] \mapsto [fg]$. If $f \sim_{x_0,y_0} g : (X, x_0) \to (Y, y_0)$, then $f_n^\sharp = g_n^\sharp$.

2.2.3. Homology

A simplex $s = \overline{\{v_0, v_1, \dots, v_m\}}$, with the vertices ordered into $v_0 v_1 \cdots v_m$, is called an oriented simplex, denoted $\overrightarrow{s} = \overrightarrow{v_0 v_1 \cdots v_m}$.

A singular n-simplex α of a topological space X is a continuous map $\alpha : \overrightarrow{s} \to X$, where $\overrightarrow{s}$ is an oriented simplex. Let $C_{-1}(X) = 0$, and let $C_n(X)$ be the free Abelian group generated by all singular n-simplexes. For $n \geq 0$, define the boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ such that, for each singular n-simplex $\alpha : \overrightarrow{v_0 v_1 \cdots v_n} \to X$, $\partial_n(\alpha) = \sum^n_{i=0} (-1)^i face_i(\alpha)$, where $face_i(\alpha) = \alpha|_{\overrightarrow{v_0 \cdots v_{i-1} v_{i+1} \cdots v_n}}$.

Since, for all $n \geq 0$, $B_n(X) = \partial_{n+1}(C_{n+1}(X))$ is a subgroup of $Z_n(X) = \{\alpha \in C_n(X) | \partial_n(\alpha) = 0\}$, we define the group $H_n(X) = Z_n(X)/B_n(X)$, which is called the nth homology group of X. Each element in $H_n(X)$ is called an n-homology class of X, and $\langle z \rangle$ represents the n-homology class of X containing $z \in Z_n(X)$.

Example 3. For $n > 0$, $H_n(\mathcal{S}^n)$ is isomorphic to the integer group Z. When $0 < m \neq n$, $H_m(\mathcal{S}^n) = 0$.

A map $f : X \to Y$ induces a homomorphism $f_n^* : H_n(X) \to H_n(Y)$, $\langle \sum \lambda_i \alpha_i \rangle \mapsto \langle \sum \lambda_i f \alpha_i \rangle$, where each α_i is a singular n-simplex. There are some facts:

- If $f \sim g$, then $f_n^* = g_n^*$.
- $(fg)_n^* = f_n^* g_n^*$.
- If $f : X \to Y$ is a homotopy equivalence, then f_n^* is an isomorphism.

(Whitehead Theorem). *Consider the polyhedron $|K|$ of an n-complex K. If it is connected and simply connected and has trivial homology groups at dimensions $1 \leq m \leq n - 1$, then $|K|$ is homotopy equivalent to W_r^n for some r.*

For any topological space X and integer $n > 1$, there is a canonical homomorphism $\mathcal{H} : \pi_n(X) \to H_n(X)$. $\mathcal{H}$ is called a Hurewicz homomorphism. It is natural in the sense that $f_n^* \mathcal{H} = \mathcal{H} f_n^\sharp$ for any continuous map f and $n > 1$.

(Hurewicz Theorem). *For any topological space X and integer $n > 1$, if X is connected and simply connected and has trivial homology groups at dimensions $1 \leq m \leq n - 1$, then $\mathcal{H} : \pi_n(X) \to H_n(X)$ is an isomorphism.*

3. Rendezvous and its signature

Definition 1. An n-rendezvous task T is defined as a triple $(K, \sigma(\Sigma^n), \varphi)$, where the decision space K is an $(n+1)$-complex, $\sigma(\Sigma^n)$ is a subdivision of the complex Σ^n, and $\varphi : \sigma(\Sigma^n) \to K$ is a simplicial embedding. The complex Σ^n is the boundary of the simplex $\nabla^{n+1} = \overline{\{v_0, v_1, \dots v_{n+1}\}}$ where $v_0 = (1, 0, \dots, 0)$, $v_1 = (0, 1, 0, \dots, 0), \dots, v_n = (0, \dots, 0, 1)$ and $v_{n+1} = (-1, -1, \dots, -1)$. We assume that $|K|$ is connected and simply connected.

For a topological illustration of rendezvous, please refer to Fig. 2.

Let $K_U = \varphi(\sigma(\overline{\{v_{i_0}, v_{i_1}, \dots, v_{i_k}\}})) \subset K$, for $U = \{i_0, i_1, \dots, i_k\} \subset \{0, \dots, n+1\}$. Let $D_{\mathbb{I}} = \{0, \dots, n+1\}$, and $D_{\mathbb{O}}$ be the set of vertices of K. For any positive integer m, the m-process task represented by the rendezvous task $(K, \sigma(\Sigma^n), \varphi)$ is $T = (\mathcal{I}, \mathcal{O}, \Delta)$, where $\mathcal{I} = (D_{\mathbb{I}} \bigcup \{\perp\})^m \setminus \{(\perp, \dots, \perp)\}$, $\mathcal{O} = (D_{\mathbb{O}} \bigcup \{\perp\})^m \setminus \{(\perp, \dots, \perp)\}$, and

$$\Delta(I) = \begin{cases} \{O | O \text{ matches } I, \text{ and } \overline{val(O)} \in K_{val(I)}\} & \text{if } val(I) \neq \{0, \dots, n+1\} \\ \{O | O \text{ matches } I, \text{ and } \overline{val(O)} \in K\} & \text{otherwise.} \end{cases} \tag{3.1}$$

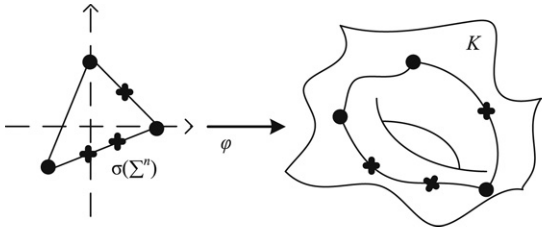

Fig. 2. A rendezvous task $(K, \sigma(\Sigma^n), \varphi)$.

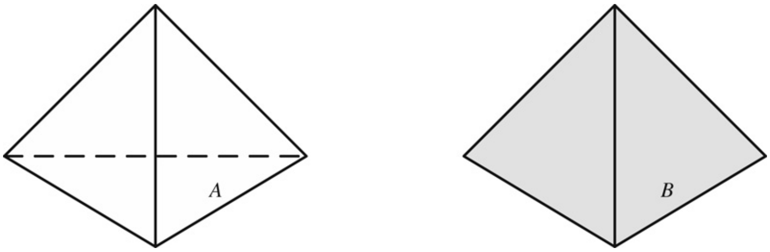

Fig. 3. (4, 3)-agreement and trivial 3-simplex agreement.

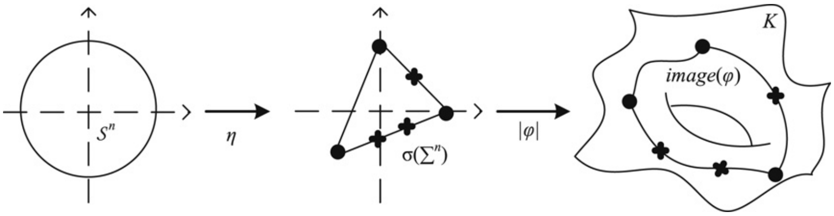

Fig. 4. An illustration of $|\varphi|\eta$.

Here $val(I)$ stands for the set of values in $D_{\mathbb{I}}$ appearing in I, and likewise for $val(O)$. Intuitively, in each execution, the processes converge on a simplex.

Example 4. The task of $(n+2, n+1)$-agreement, which means the set agreement whose set of input values is $\{0, \ldots, n+1\}$ and each of whose executions produces at most $n+1$ distinct values, is described by the rendezvous task $(\Sigma^n, \Sigma^n, id_{\Sigma^n})$.

Example 5. The task of trivial $(n+1)$-simplex agreement, where the input data type is $\{0, 1, \ldots, n+1\}$ and the output values in each execution are among the assigned input values, is described by the rendezvous task $(\nabla^{n+1}, \Sigma^n, i_{\Sigma^n})$, with i_{Σ^n} denoting the inclusion map from Σ^n to ∇^{n+1}.

The two tasks are illustrated in Fig. 3, with Σ^2 and the embedding maps omitted. The left one is a hollow tetragon, representing (4, 3)-agreement, while the right one is solid, representing trivial 3-simplex agreement.

Definition 2. An n-dimensional rendezvous task $(K, \sigma(\Sigma^n), \varphi)$ is said to be *nice* if

$$H_q(|K|) = \begin{cases} \text{a free Abelian group} & \text{if } q = n \text{ or } q = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Example 6. Both $(n+2, n+1)$-agreement and $(n+1)$-simplex agreement are nice n-rendezvous tasks.

Hereunder, the symbol τ stands for a fixed generator of $H_n(\mathcal{S}^n) \cong Z$, and η represents a fixed homeomorphism from $\mathcal{S}^n$ to $|\Sigma^n|$.

Consider an arbitrary rendezvous task $T = (K, \sigma(\Sigma^n), \varphi)$. Since $|\sigma(\Sigma^n)| = |\Sigma^n|$, the map $\eta : \mathcal{S}^n \to |\sigma(\Sigma^n)|$ makes sense. And $(|\varphi|\eta)_n^*(\tau)$ is an element in $H_n(|K|)$. See Fig. 4 for an illustration of $|\varphi|\eta$. We are ready to assign T an algebraic signature, denoted $sig(T)$.

Definition 3. $sig(T) = (H_n(|K|), (|\varphi|\eta)_n^*(\tau))$ for rendezvous task $T = (K, \sigma(\Sigma^n), \varphi)$.

Example 7. The signature of $(n+2, n+1)$-agreement is $(H_n(\Sigma^n), \eta_n^*(\tau))$, where $H_n(\Sigma^n)$ is isomorphic to the group of integers, with $\eta_n^*(\tau)$ as a generator.

Example 8. The signature of trivial $(n+1)$-simplex agreement is $(\{0\}, 0)$.

Given topological spaces X, Y, and A, a map $f : X \to Y$ is said to be a map from $(X, \ \varphi : A \to X)$ to $(Y, \ \varphi' : A \to Y)$ if it satisfies that $f\varphi = \varphi'$.

Given groups G, G', and $e \in G$, $e' \in G'$, then (G, e) and (G', e') are said to be homomorphic (respectively, isomorphic) if there is a homomorphism (respectively, isomorphism) $h : G \to G'$ such that $h(e) = e'$. And h is said to be a homomorphism (respectively, isomorphism) from (G, e) to (G', e').

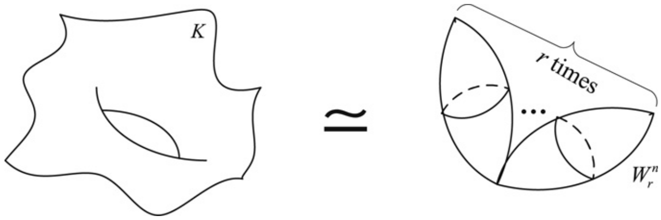

Fig. 5. An illustration of Lemma 3.

4. Necessity for general rendezvous

Arbitrarily choose two rendezvous tasks $T = (K, \sigma(\Sigma^n), \varphi)$ and $T' = (K', \sigma'(\Sigma^n), \varphi')$. In this section, we claim that, if T implements T', then there is a homomorphism from $sig(T)$ to $sig(T')$. The reasoning process includes two parts: on the one hand, the implementability implies the existence of a map. On the other hand, the existence of a map implies a homomorphism.

Lemma 1. *T implements T' if and only if there is a map from $(|K|, |\varphi|)$ to $(|K'|, |\varphi'|)$.*

Proof. The proof of Lemma 6.7 in [11] is independent of dimension, so it can be adapted to our case without any modification. We omit the proof here. □

Lemma 2. *A map from $(|K|, |\varphi|)$ to $(|K'|, |\varphi'|)$ induces a homomorphism from $(H_n(|K|), (|\varphi|\eta)_n^*(\tau))$ to $(H_n(|K'|), (|\varphi'|\eta)_n^*(\tau))$.*

Proof. This is well known in topology. □

Corollary 1. *Rendezvous task $T = (K, \sigma(\Sigma^n), \varphi)$ implements $T' = (K', \sigma'(\Sigma^n), \varphi')$ only if there is a homomorphism from $sig(T)$ to $sig(T')$.*

Example 9. The signatures of $(n+2, n+1)$-agreement and trivial $(n+1)$-simplex agreement are $(Z, 1)$ and $(\{0\}, 0)$ up to isomorphism, respectively, where Z is the group of integers. Of course there is no homomorphism from $(\{0\}, 0)$ to $(Z, 1)$. As a result, one cannot implement $(n+2, n+1)$-agreement from trivial $(n+1)$-simplex agreement.

5. Sufficiency for nice rendezvous

It is known that, for general n-rendezvous tasks $(K, \sigma(\Sigma^n), \varphi)$ and $(K', \sigma'(\Sigma^n), \varphi')$, the inverse of Lemma 2 does not hold. However, if both of them are nice, then it does hold, so the implementability of one nice rendezvous task from another is completely determined by their signatures. To prove this fact, we first show that $|K|$ and $|K'|$ are homotopy equivalent to the wedge sums of some n-spheres, and then show that, for two wedge sums W_p^n and W_q^n of n-spheres, any homomorphism from $H_n(W_p^n)$ and $H_n(W_q^n)$ can be induced by a continuous map from W_p^n to W_q^n. These two aspects eventually lead to the desired map from $(|K|, |\varphi|)$ to $(|K'|, |\varphi'|)$. By default, all the rendezvous tasks discussed in this section are of dimension higher than 1.

Lemma 3. *Given a nice rendezvous task $(K, \sigma(\Sigma^n), \varphi)$, there is a positive integer r such that $|K| \simeq W_r^n$, where W_r^n is the wedge sum of r n-spheres.*

Proof. The proof can be figured out by following the techniques used in the proof of Whitehead Theorem, so it is omitted here. □

The meaning of Lemma 3 is illustrated in Fig. 5.

Now we have to set some notations. For a natural number r, let $\mathcal{N}_r = \{1, 2, \ldots, r\}$. Recall that $W_r^n = (\mathcal{S}^n \times \mathcal{N}_r)/f_r$, where $f_r : \{v_0\} \times \mathcal{N}_r \to *$ with $v_0 = (1, 0, \ldots, 0) \in \mathcal{S}^n$. Let $\rho_r : \mathcal{S}^n \times \mathcal{N}_r \to W_r^n$ be the quotient map. For any positive integer $i \le r$, let $\theta_i : \mathcal{S}^n \to \mathcal{S}^n \times \mathcal{N}_r, v \mapsto (v, i)$. $H_n(W_r^n)$ is a free Abelian group generated by r generators $\{\alpha_1, \ldots, \alpha_r\}$, where $\alpha_i = (\rho_r\theta_i)_n^*(\tau)$. Define $\varrho_i : W_r^n \to \mathcal{S}$,

$$\rho_r(v, j) \mapsto \begin{cases} v & \text{if } j = i \\ v_0 & \text{otherwise.} \end{cases} \tag{5.1}$$

An important property of ϱ_i is that $\alpha = \Sigma_1^r \lambda_i\alpha_i \in H_n(W_r^n)$ if and only if $(\varrho_i)_n^*(\alpha) = \lambda_i\tau$ for all $1 \le i \le r$.

Lemma 4. *Given wedge sums W_p^n and W_q^n of n-spheres, any homomorphism from $H_n(W_p^n)$ to $H_n(W_q^n)$ can be induced by a continuous map from W_p^n to W_q^n.*

Proof. Consider an arbitrary homomorphism $h : H_n(W_p^n) \to H_n(W_q^n)$, and suppose, for $1 \le i \le p$, $h(\alpha_i) = \sum_{j=1}^q \lambda_{i,j}\alpha_j$. Now we construct a map which induces h. Fix some i.

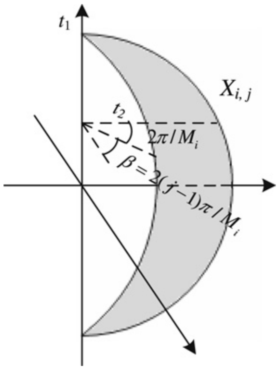

Fig. 6. An illustration of $X_{i,j}$.

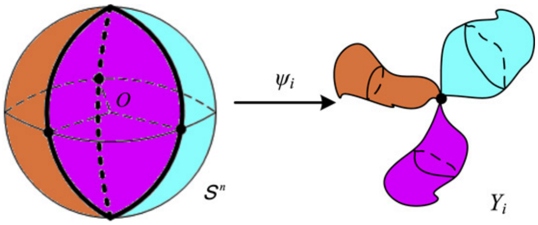

Fig. 7. The illustration of ψ_i.

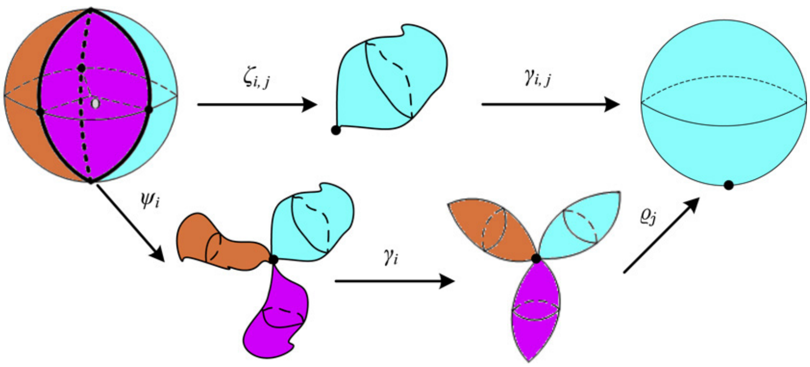

Fig. 8. The diagram for ϱ_j, γ_i, ψ_i, $\gamma_{i,j}$, and $\zeta_{i,j}$.

Let $M_i = \sum_{j=1}^{q} |\lambda_{i,j}|$, and $k_i : \mathcal{N}_{M_i} \to \mathcal{N}_q$ be such that $\sum_{l=1}^{k_i(j)-1} |\lambda_{i,l}| + 1 \le j \le \sum_{l=1}^{k_i(j)} |\lambda_{i,l}|$ for all $j \in \mathcal{N}_{M_i}$. It always holds that $\lambda_{i,k_i(j)} \neq 0$.

Recall that $\mathcal{S}^n = \{(t_0, \ldots, t_{n-2}, t_{n-1}\sin\beta, t_{n-1}\cos\beta) \in R^{n+1} | t_{n-1} \ge 0, 2\pi \ge \beta \ge 0, \sum_{j=0}^{n-1} t_j^2 = 1\}$. For $1 \le j \le M_i$, let $X_{i,j}$ be the subspace of $\mathcal{S}^n$ with $2(j-1)\pi/M_i \le \beta \le 2j\pi/M_i$, as illustrated in Fig. 6. Let $X_i = \bigcup_{j=1}^{M_i} \partial X_{i,j}$, where $\partial X_{i,j}$ is the boundary of $X_{i,j}$. Consider the quotient space $Y_i = \mathcal{S}^n/\xi_i$, where $\xi_i : X_i \to *$. Let $\psi_i : \mathcal{S}^n \to Y_i$ be the quotient map. See Fig. 7. Note the facts:

- $\psi_i(X_{i,j})$ is homeomorphic to $\mathcal{S}^n$.
- $\psi_i|_{\dot{X}_{i,j}} : \dot{X}_{i,j} \to \psi_i(X_{i,j}) - \psi_i(v_0)$ is a homeomorphism, where $\dot{X}_{i,j}$ is the interior of $X_{i,j}$.
- Y_i is the wedge sum of all $\psi_i(X_{i,j})$ at $\psi_i(v_0)$, $1 \le j \le M_i$.

Assume $\gamma_{i,j} : \psi_i(X_{i,j}) \to \mathcal{S}^n$ to be a homeomorphism. Without loss of generality, assume $\gamma_{i,j}\psi_i(v_0) = v_0$. Define $\zeta_{i,j} : \mathcal{S}^n \to \psi_i(X_{i,j})$,

$$v \mapsto \begin{cases} \psi_i(v) & \text{if } v \in X_{i,j} \\ \psi_i(v_0) & \text{otherwise.} \end{cases} \tag{5.2}$$

Then $\gamma_{i,j}\zeta_{i,j} : \mathcal{S}^n \to \mathcal{S}^n$ satisfies that $\gamma_{i,j}\zeta_{i,j}|_{\dot{X}_{i,j}} = \gamma_{i,j}|_{\psi_i(X_{i,j})-\psi_i(v_0)}\psi_i|_{\dot{X}_{i,j}} : \dot{X}_{i,j} \to \mathcal{S}^n - v_0$ is a homeomorphism, and $\gamma_{i,j}\zeta_{i,j}(\mathcal{S}^n - \dot{X}_{i,j}) = \{v_0\}$. See the top part of Fig. 8. One can show that $\gamma_{i,j}\zeta_{i,j}$ is homotopic to a homeomorphism $g : \mathcal{S}^n \to \mathcal{S}^n$. So, $\gamma_{i,j}$ can be properly chosen such that $(\gamma_{i,j}\zeta_{i,j})_n^* = id_{H_n(\mathcal{S}^n)}$. For each $1 \le j \le M_i$, choose $\gamma_{i,j}$ in this way.

$$\begin{tikzcd}
S^n \times \mathcal{N}_{M_i} \arrow[r, "\delta_i"] \arrow[d, "\rho_{M_i}"] & S^n \times \mathcal{N}_q \arrow[d, "\rho_q"] \\
W^n_{M_i} \arrow[r, "\iota_i"] & W^n_q
\end{tikzcd}$$

Fig. 9. An illustration of ι_i.

$$\begin{tikzcd}
S^n \times \mathcal{N}_p \arrow[r, "\omega = \prod \omega_i"] \arrow[d, "\rho_p"] & \bigsqcup_{i=1}^p W^n_{M_i} \arrow[d, "\rho"] \arrow[dr, "\iota"] & \\
W^n_p \arrow[r, "\mu"] & W^n_M \arrow[r, "\delta"] & W^n_q
\end{tikzcd}$$

Fig. 10. The diagram for ω, ι, ρ, μ, and δ.

$$\begin{tikzcd}
W^n_p \arrow[r, "\mu"] \arrow[dr, "f"'] & \bigsqcup_{i=1}^p W^n_{M_i} \arrow[d, "\delta"] \\
& W^n_q
\end{tikzcd}$$

Fig. 11. An illustration of the map f.

Now, define $\gamma_i : Y_i \to W^n_{M_i}$,

$$\psi_i(v) \mapsto \rho_{M_i}(\gamma_{i,j}(\psi_i(v)), j) \quad \text{if } v \in X_{i,j}. \tag{5.3}$$

An important property of γ_i is that, for any $1 \le j \le M_i$, $\varrho_j \gamma_i \psi_i = \gamma_{i,j}\zeta_{i,j}$. See Fig. 8. Hence, for $1 \le j \le M_i$, $(\varrho_j)^*_n(\gamma_i\psi_i)^*_n(\tau) = (\varrho_j\gamma_i\psi_i)^*_n(\tau) = (\gamma_{i,j}\zeta_{i,j})^*_n(\tau) = id_{H_n(\mathcal{S}^n)}(\tau) = \tau$, which means that $(\gamma_i\psi_i)^*_n(\tau) = \sum_{j=1}^{M_i} \alpha_j$.

Define $\delta_i : \mathcal{S}^n \times \mathcal{N}_{M_i} \to \mathcal{S}^n \times \mathcal{N}_q$,

$$(t_0, \ldots, t_n, j) \mapsto \begin{cases} (t_0, \ldots, t_n, k_i(j)) & \text{if } \lambda_{i,k_i(j)} \ge 0 \\ (t_0, \ldots, t_{n-1}, -t_n, k_i(j)) & \text{otherwise.} \end{cases} \tag{5.4}$$

Then there is a unique $\iota_i : W^n_{M_i} \to W^n_q$ such that $\rho_q\delta_i = \iota_i\rho_{M_i}$. See Fig. 9. For $1 \le j \le M_i$ and $1 \le l \le q$, consider $(\varrho_l)^*_n(\iota_i)^*_n(\alpha_j) = (\varrho_l\iota_i)^*_n(\alpha_j) = (\varrho_l\iota_i\rho_{M_i}\theta_j)^*_n(\tau) = (\varrho_l\rho_q\delta_i\theta_j)^*_n(\tau)$

$$= \begin{cases} (id_{\mathcal{S}^n})^*_n(\tau) & \text{if } l = k_i(j) \text{ and } \lambda_{i,k_i(j)} > 0 \\ (id'_{\mathcal{S}^n})^*_n(\tau) & \text{if } l = k_i(j) \text{ and } \lambda_{i,k_i(j)} < 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

where $id'_{\mathcal{S}^n} : (t_0, \ldots, t_n) \mapsto (t_0, \ldots, t_{n-1}, -t_n)$. Because $(id'_{\mathcal{S}^n})^*_n(\tau) = -\tau$,

$$(\iota_i)^*_n(\alpha_j) = \begin{cases} \alpha_{k_i(j)} & \text{if } \lambda_{i,k_i(j)} > 0 \\ -\alpha_{k_i(j)} & \text{otherwise.} \end{cases} \tag{5.6}$$

Now, let $M = \sum_{i=1}^p M_i$, and $\rho : \bigsqcup_{i=1}^p W^n_{M_i} \to W^n_M$ be the obvious quotient map, i.e. for any $y = \rho_i(v, j)$, $\rho(y) = \rho_M(v, \sum_{l=1}^{i-1} M_l + j)$. Here, $\bigsqcup$ stands for disjoint union.

Let $\omega_i = \gamma_i\psi_i$. Define $\omega = \prod_{i=1}^p \omega_i : \mathcal{S}^n \times \mathcal{N}_p \to \bigsqcup_{i=1}^p W^n_{M_i}$, $(v, i) \mapsto \omega_i(v) \in W^n_{M_i}$. We have $\omega\theta_i = \omega_i$. Then there is a unique map $\mu : W^n_p \to W^n_M$ such that $\mu\rho_p = \rho\omega$. See the left part of Fig. 10.

Define $\iota : \bigsqcup_{i=1}^p W^n_{M_i} \to W^n_q$, which satisfies $\iota|_{W^n_{M_i}} = \iota_i$ for each $1 \le i \le p$. There is a unique map $\delta : W^n_M \to W^n_q$ satisfying $\delta\rho = \iota$. See the right part of Fig. 10.

Let $f = \delta\mu : W^n_p \to W^n_q$. See Fig. 11. We proceed to show that $f^*_n = h$. For any $1 \le i \le p$, $f^*_n(\alpha_i) = (\delta\mu)^*_n(\alpha_i) = (\delta\mu\rho_p\theta_i)^*_n(\tau) = (\iota\omega\theta_i)^*_n(\tau) = (\iota_i\omega_i)^*_n(\tau) = (\iota_i\gamma_i\psi_i)^*_n(\tau) = (\iota_i)^*_n(\gamma_i\psi_i)^*_n(\tau) = (\iota_i)^*_n(\sum_{j=1}^{M_i}\alpha_j) = \sum_{j=1}^{M_i}(\iota_i)^*_n(\alpha_j) = \sum_{j=1}^{M_i} d_{i,j}\alpha_{k_i(j)}$, where

$$d_{i,j} = \begin{cases} 1 & \text{if } \lambda_{i,k_i(j)} > 0 \\ -1 & \text{if } \lambda_{i,k_i(j)} < 0. \end{cases} \tag{5.7}$$

As a result, $f^*_n(\alpha_i) = \sum_{j=1}^q \lambda_{i,j}\alpha_j$, which means that $f^*_n = h$. $\square$

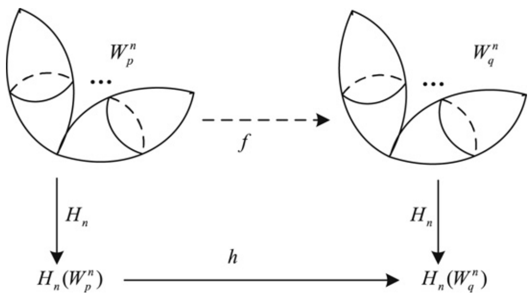

Fig. 12. The homomorphism h is induced by the map f.

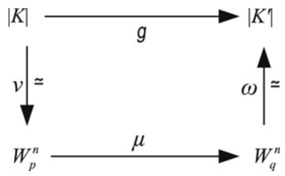

Fig. 13. The maps ν, ω, μ, and g.

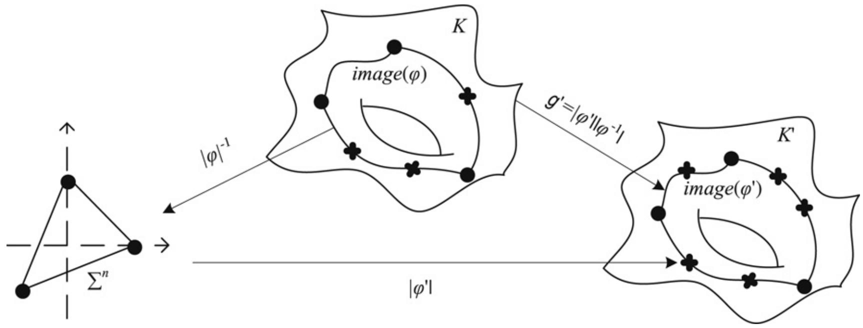

Fig. 14. The continuous map g'.

Lemma 4 is illustrated in Fig. 12.

Lemma 5. *Given nice rendezvous tasks $T = (K, \sigma(\Sigma^n), \varphi)$ and $T' = (K', \sigma'(\Sigma^n), \varphi')$, any homomorphism from $sig(T)$ to $sig(T')$ can be induced by a map from $(|K|, |\varphi|)$ to $(|K'|, |\varphi'|)$.*

Proof. Let $sig(T) = (G, e)$ and $sig(T') = (G', e')$, i.e. $e = (|\varphi|\eta)_n^*(\tau)$, $e' = (|\varphi'|\eta)_n^*(\tau)$. Consider a homomorphism $h : (G, e) \to (G', e')$. We proceed to construct a map from (K, φ) to (K', φ') which induces h. According to Lemma 3, there are integers p and q, and maps $\nu : |K| \to W_p^n$ and $\omega : W_q^n \to |K'|$, which induce isomorphisms $\nu_n^* : H_n(|K|) \to H_n(W_p^n)$ and $\omega_n^* : H_n(W_q^n) \to H_n(|K'|)$. Let $\varepsilon = \nu_n^*(e)$ and $\varepsilon' = (\omega_n^*)^{-1}(e')$. Let $h' = (\omega_n^*)^{-1} h (\nu_n^*)^{-1} : (H_n(W_p^n), \varepsilon) \to (H_n(W_q^n), \varepsilon')$.

By Lemma 4, there is a continuous map $\mu : W_p^n \to W_q^n$, such that $\mu_n^* = h'$. Let $g = \omega\mu\nu : |K| \to |K'|$. See Fig. 13. We have $g_n^* = \omega_n^* \mu_n^* \nu_n^* = \omega_n^* h' \nu_n^* = h$. Now, we show that there is a map from $(|K|, |\varphi|)$ to $(|K'|, |\varphi'|)$ which is homotopical to g.

Let $A = image(\varphi)$ and $\tilde{\varphi} : \sigma(\Sigma^n) \to A, s \mapsto \varphi(s)$. Then $|\tilde{\varphi}|^{-1}$ exists because φ is an embedding. Define $g' = |\varphi'||\tilde{\varphi}|^{-1} : A \to |K'|$. g' is illustrated in Fig. 14.

By Lemma 3 and the Hurewicz Theorem, the Hurewicz homomorphism $\mathcal{H} : \pi_n(|K'|) \to H_n(|K'|)$ is an isomorphism. It follows from $g_n^*(e) = e'$ that $\mathcal{H}^{-1} g_n^*(e) = \mathcal{H}^{-1}(e')$.

On the one hand, $\mathcal{H}^{-1} g_n^*(e) = \mathcal{H}^{-1} g_n^*(|\varphi|\eta)_n^*(\tau) = \mathcal{H}^{-1}(g|\varphi|\eta)_n^*(\tau) = (g|\varphi|\eta)_n^\sharp \mathcal{H}^{-1}(\tau)$. Because $\mathcal{H} : \pi_n(\mathcal{S}^n) \to H_n(\mathcal{S}^n)$ is also an isomorphism and τ is a generator of $H_n(\mathcal{S}^n)$, $\mathcal{H}^{-1}(\tau) = d[id_{\mathcal{S}^n}]$, where d is either 1 or -1. Then, $\mathcal{H}^{-1} g_n^*(e) = (g|\varphi|\eta)_n^\sharp \mathcal{H}^{-1}(\tau) = (g|\varphi|\eta)_n^\sharp (d[id_{\mathcal{S}^n}]) = d[g|\varphi|\eta]$. On the other hand, we can show that $\mathcal{H}^{-1}(e') = d[|\varphi'|\eta]$. As a result, $[g|\varphi|\eta] = [|\varphi'|\eta] \in \pi_n(|K'|)$, which means that $g|\varphi|\eta \sim |\varphi'|\eta$. Hence, $g|\varphi| \sim |\varphi'|$, $g|_A|\tilde{\varphi}| \sim |\varphi'|$, and $g|_A \sim |\varphi'||\tilde{\varphi}|^{-1} = g'$.

According to the Homotopic Extension Theorem, there is a continuous map $f : |K| \to |K'|$ which satisfies that $f \sim g$ and $f|_A = g'$. See Fig. 15. Then we have two facts. First, $f_n^* = g_n^* = h$. Second, $f|\varphi| = f|_A|\tilde{\varphi}| = g'|\tilde{\varphi}| = |\varphi'|$. Altogether, f is a map from $(|K|, |\varphi|)$ to $(|K'|, |\varphi'|)$ which induces h. □

Theorem 1. *Given nice rendezvous tasks $T = (K, \sigma(\Sigma^n), \varphi)$ and $T' = (K', \sigma'(\Sigma^n), \varphi')$, T implements T' if and only if there is a homomorphism from $sig(T)$ to $sig(T')$.*

Proof. The necessity follows from Corollary 1. The sufficiency follows from Lemma 1 and Lemma 5. □

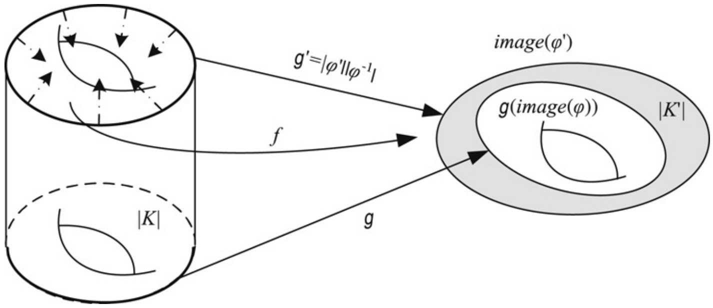

Fig. 15. The extension f of g' is homotopic to g.

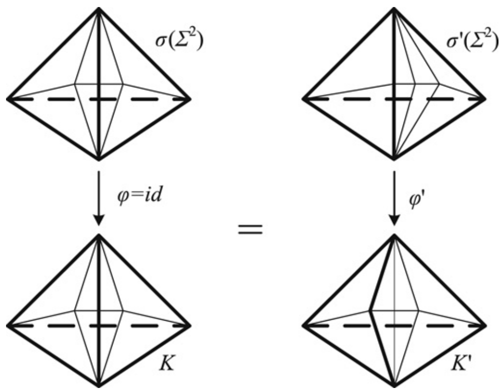

Fig. 16. The two different tasks are equivalent because $image(\varphi) = image(\varphi')$.

6. Examples and further facts

In this section, some facts are presented to illustrate the power of Theorem 1. By default, all the tasks in the following are n-rendezvous, for some fixed $n > 1$.

Implementability induces an equivalence relation $\mathcal{R} = \{(T, T')|$ the tasks T and T' can implement each other$\}$ among rendezvous tasks. We use $\widehat{T}$ to denote the equivalence class containing task T.

The following proposition claims that the power of a nice rendezvous task $(K, \sigma(\Sigma^n), \varphi)$ depends on $image(\varphi)$, rather than on σ or φ.

Proposition 1. *Given two nice rendezvous tasks* $T = (K, \sigma(\Sigma^n), \varphi)$ *and* $T' = (K, \sigma'(\Sigma^n), \varphi')$, $\widehat{T} = \widehat{T'}$ *if* $image(\varphi) = image(\varphi')$.

Proof. Let $X = image(|\varphi|)$ and $i_X : X \to |K|,\ x \mapsto x$. For any map $f : A \to B$, we use $\tilde{f}$ to denote the map $\tilde{f} : A \to image(f),\ x \mapsto f(x)$. On the one hand, $(|\varphi|\eta)^*_n(\tau) = (i_X\widetilde{|\varphi|}\eta)^*_n(\tau) = (i_X)^*_n(\widetilde{|\varphi|}\eta)^*_n(\tau)$. On the other hand, $(|\varphi'|\eta)^*_n(\tau) = (i_X)^*_n(\widetilde{|\varphi'|}\eta)^*_n(\tau)$. Because $\widetilde{|\varphi'|}\widetilde{|\varphi|}^{-1} : X \to X$ is a homeomorphism and $H_n(X) = H_n(\mathcal{S}^n) = Z$, $(\widetilde{|\varphi'|}\widetilde{|\varphi|}^{-1})^*_n = k \cdot id_{H_n(X)}$, where k is either 1 or -1. Then, $(\widetilde{|\varphi'|}\eta)^*_n(\tau) = (\widetilde{|\varphi'|}\widetilde{|\varphi|}^{-1}\widetilde{|\varphi|}\eta)^*_n(\tau) = (\widetilde{|\varphi'|}\widetilde{|\varphi|}^{-1})^*_n(\widetilde{|\varphi|}\eta)^*_n(\tau) = k(\widetilde{|\varphi|}\eta)^*_n(\tau)$. As a result, we have $(|\varphi'|\eta)^*_n(\tau) = k(|\varphi\eta|)^*_n(\tau)$.

There is an isomorphism $h : (H_n(|K|), (|\varphi|\eta)^*_n(\tau)) \to (H_n(|K|), (|\varphi'|\eta)^*_n(\tau)),\ \alpha \mapsto k\alpha$. It follows from Theorem 1 that $\widehat{T} = \widehat{T'}$. □

Proposition 1 is illustrated by Fig. 16. In this figure, nice 2-rendezvous tasks $T = (K, \sigma(\Sigma^2), \varphi)$ and $T' = (K, \sigma'(\Sigma^2), \varphi')$ are presented. σ and σ' are different; the dark lines illustrate the 1-skeleton of Σ^2. The dark lines in K and K' respectively illustrate the images of the 1-skeleton of Σ^2 under φ and φ'. T and T' stand for two different tasks, but, according to Proposition 1, they are equivalent.

Proposition 2. *For any integer* $m \geq 1$, *there is a nice rendezvous task* T_m *with the signature* (Z, m), *where* Z *is the group of integers. And* $\widehat{T_m} \neq \widehat{T_{m'}}$ *for* $m \neq m'$.

Proof. Because the signature of $(n+2, n+1)$-agreement is isomorphic to $(Z, 1)$, let T_1 stand for $(n+2, n+1)$-agreement. Now go on with the case where $m \geq 2$. The proof is done in three steps.

First, consider the standard n-sphere $\mathcal{S}^n$. For any integer $m \geq 2$, choose a subspace $X \subseteq \mathcal{S}^n$ such that X is the union of some $(n-1)$-spheres and $\mathcal{S}^n \setminus X$ is the disjoint union of m open n-disks. Arbitrarily orient $\mathcal{S}^n$, and orient each component (which is an n-disk) of $\mathcal{S}^n \setminus X$ with the inherited orientation. Define $f : \mathcal{S}^n \to \mathcal{S}^n$, such that $image(f|_X) = \{v_0\}$, and $f|_C : C \to \mathcal{S}^n \setminus \{v_0\}$ is an orientation-preserving homeomorphism, for each connected component C of $\mathcal{S}^n \setminus X$. Obviously, $\mathcal{S}^n/f \cong \mathcal{S}^n$.

Second, consider space $Y = [0, 1] \times \mathcal{S}^n$. Choose $X \subset \{0\} \times \mathcal{S}^n$ as in step 1, and accordingly define $f : \{0\} \times \mathcal{S}^n \to \mathcal{S}^n$. It can be proven that $Y/f \simeq \mathcal{S}^n$, and the proof is omitted here. Let $\rho : Y \to Y/f$ be the quotient map.

Third, triangulate the topological space Y/f into a complex K, such that there is a subdivision $\sigma(\Sigma^n)$ of Σ^n and a simplicial embedding $\varphi : \sigma(\Sigma^n) \to K$ satisfying that $image(|\varphi|) = image(\rho|_{\{1\}\times\mathcal{S}^n})$. Consider the rendezvous task $T_m = (K, \sigma(\Sigma^n), \varphi)$. Since $|K| = Y/f \simeq \mathcal{S}^n$, T_m is a nice n-rendezvous task. Furthermore, $H_n(|K|)$ is isomorphic to Z, and $(|\varphi|\eta)^*_n(\tau) = m\alpha$, where α is a generator of $H_n(|K|)$. As a result, $sig(T_m) = (Z, m)$ up to isomorphism.

Because there is a homomorphism from (Z, m) to (Z, m') if and only if $m|m'$, $T_m \neq T_{m'}$ for $m \neq m'$. □

Proposition 3. *The following statements hold.*

- *There are infinitely many classes of rendezvous tasks.*
- *There are infinitely many classes $\{c_1, c_2, \dots\}$ of rendezvous tasks such that c_i cannot implement c_j for $i \neq j$.*
- *There is an infinite hierarchy $c_1 c_2 \cdots$ of classes of rendezvous tasks such that c_i can implement c_{i+1}, but not vice versa, for $i \geq 1$.*

Proof. To prove the first two statements, let $c_i = \widehat{T_{p_i}}$, where p_i is the ith smallest prime number. Because there is a homomorphism from (Z, i) to (Z, j) if and only if $i|j$, c_i cannot implement c_j for $i \neq j$.

To prove the last statement, let $c_i = \widehat{T_{2^i}}$. Of course, c_i can implement c_{i+1}, but not vice versa, for $i \geq 1$. □

Let T_0 be the trivial $(n + 1)$-simplex agreement.

Proposition 4. *The set $\{\widehat{T_m}|m \geq 0\}$ exhausts the equivalence classes of nice n-rendezvous tasks.*

Proof. Arbitrarily choose a nice n-rendezvous task T with $sig(T) = (G, e)$. The rest of the proof is divided into two cases.

Case 1: $e = 0$. Then there are homomorphisms both from (G, e) to $(0, 0)$ and from $(0, 0)$ to (G, e), so $T \in \widehat{T_0}$.

Case 2: $e \neq 0$. Then e has the form $e = \sum_{i=1}^{l} \lambda_i \alpha_i$, where $\{\alpha_i|1 \leq i \leq l\}$ is a set of generators of G, and there is at least one i such that $\lambda_i \neq 0$.

Now we claim that, if the generators are properly chosen, there can be only one i such that $\lambda_i \neq 0$. We prove this claim by induction on $d = |\{1 \leq i \leq l|\lambda_i \neq 0\}|$.

Step 1. If $d = 1$, the claim trivially holds.

Step 2. Hypothesize that the claim holds for all $d < j$, where $2 \leq j \leq l$, and we show that it holds for $d = j$. Without loss of generality, assume that $\lambda_1, \lambda_2 \neq 0$. There must be integers $k_1, k_2, k_3, \lambda'_1$, and λ'_2 such that $\lambda_i = k_3\lambda'_i$ for $i = 1, 2$, and $\sum_{i=1,2} k_i\lambda'_i = 1$.

Let $\alpha'_1 = \sum_{i=1,2} \lambda'_i\alpha_i$, $\alpha'_2 = -k_2\alpha_1 + k_1\alpha_2$, and $\alpha'_i = \alpha_i$ for $3 \leq i \leq l$. Because $k_1\alpha'_1 - \lambda'_2\alpha'_2 = \alpha_1$ and $k_2\alpha'_1 + \lambda'_1\alpha'_2 = \alpha_2$, $\{\alpha'_i|1 \leq i \leq l\}$ is also a set of generators of G. Then $e = \sum_{i=0}^{l} \lambda_i\alpha_i = k_3\alpha'_1 + \sum_{i=3}^{l} \lambda_i\alpha'_i$. Because $|\{i|\lambda_i \neq 0,\ i \geq 3\} \cup \{1\}| = j - 1 < j$, the claim holds by the induction hypothesis.

Thus, we assume that the generators $\{\alpha_i|0 \leq i \leq l\}$ are such that $e = k\alpha_0$ for some integer k. We can construct homomorphisms $h : (G, e) \to (Z, k)$, $\sum_{i=0}^{l} \lambda_i\alpha_i \mapsto \lambda_0$, and $h' : (Z, k) \to (G, e)$, $\lambda \mapsto \lambda\alpha_0$. As a result, $T \in \widehat{T_k}$.

To sum, in either case, there always exists $k \geq 0$ such that $T \in \widehat{T_k}$. □

Proposition 5. *$(n+2, n+1)$-agreement can implement any nice n-rendezvous task, and any n-rendezvous task can implement trivial $(n + 1)$-simplex agreement.*

Proof. The signature of $(n + 2, n + 1)$-agreement is isomorphic to $(Z, 1)$. Assume (G, e) to be the signature of a nice n-rendezvous task T. The map $h : k \mapsto ke$ is a homomorphism from $(Z, 1)$ to (G, e), so $(n+2, n+1)$-agreement implements T.

The second part of this proposition follows from this fact: the trivial $(n + 1)$-simplex agreement can be wait-freely implemented from read/write registers (each process simply outputs its private input value). □

7. Conclusion

This paper completely characterizes the computational power of *nice* rendezvous tasks of arbitrary dimension by their signatures. Intuitively, niceness of an n-rendezvous task means nonexistence of holes in its decision space, except at dimension n. Despite this connectivity constraint, nice rendezvous includes many interesting tasks such as set agreement, simplex agreement, and approximation agreement. Thus, this work also provides an insight into the computational power of general rendezvous tasks.

Our main techniques come from algebraic topology. Generally, homology groups are conceptually simpler than homotopy groups; they are easier to calculate and even are computable for finite simplicial complexes. Hence, we make an effort to bypass homotopy groups in formulating the definitions and presenting the results, including the lemmas, the propositions, and the theorem. Also, we use homology groups in the reasoning process whenever possible. The only exception is the proof of Lemma 5, where homotopy groups are still involved. This involvement may be necessary, because it is hard to derive homotopy relation of maps only through homology theory.

For a more comprehensive understanding of general rendezvous, it is interesting to study the following issues. First, how much can the niceness requirement be relaxed, without invalidating Theorem 1? Second, exactly which rendezvous tasks break the equivalence between signatures and computational power? Third, how can the notion of rendezvous be generalized so as to include more significant tasks such as renaming and consensus?

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